A partial characterization of the core in Bertrand oligopoly TU-games with transferable technologies

Aymeric Lardon

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Abstract

In this article we study Bertrand oligopoly TU-games with transferable technologies under the α and β-approaches (Aumann 1959). Although the convexity property does not always hold, we show that it is satisfied when firms' marginal costs are not too heterogeneous. Furthermore, we prove that the core of any game can be partially characterized by associating a Bertrand oligopoly TU-game derived from the most efficient technology. Such a game turns to be an efficient convex cover (Rulnick and Shapley 1997) of the original one. This result implies that the core is non-empty and contains a subset of payoff vectors with a symmetric geometric structure easy to compute.

Keywords: Bertrand oligopoly TU-games; Transferable technologies; Core; Convexity property;
JEL Classifications: C71, D43

1 Introduction

The literature on cooperative games uses three main typologies in order to classify oligopoly TU (Transferable Utility)-games.
The first typology concerns competition type namely Cournot (quantity) and Bertrand (price) competition in which firms propose homogeneous or differentiated products.
The second typology distinguishes two types of oligopolies according to the possibility or not for cooperating firms to transfer their technologies.[1] When technologies are transferable, cooperating firms are allowed to produce according to the most efficient technology available in the cartel. When technologies are not transferable such a transfer is not possible.
The third typology concerns blocking rules used to convert a normal form oligopoly game into an oligopoly TU-game. Among many blocking rules we can cite three main approaches. The first two called the α and β-approaches are suggested by Aumann (1959).

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[1]We refer to Norde et al. (2002) for a detailed discussion on this typology.

According to the first, every cartel computes its max-min profit, i.e. the profit it can guarantee regardless of what outsiders do. The second approach consists in computing the min-max cartel profit, i.e. the minimal profit for which external firms can prevent the cartel from getting more. The third approach called the γ-approach is proposed by Chander and Tulkens (1997). The cartel profit is then computed by considering a competition setting in which the cartel faces external firms acting individually.

When technologies are not transferable, Zhao (1999b) proves that the core of Cournot oligopoly TU-games under the α and β-approaches is non-empty if every individual profit function is continuous and concave.[2] Furthermore, Norde et al. (2002) show that such games are convex when the inverse demand function and cost functions are linear, and Driessen and Meinhardt (2005) provide sufficient conditions to guarantee the convexity property in a more general setting. Under the γ-approach, Lardon (2012) shows that Cournot oligopoly TU-games are balanced if every individual profit function is concave and provides a single-valued allocation rule in the core, called NP(Nash Pro rata)-value, when cost functions are linear. Concerning price competition where firms operate at a constant and identical marginal cost, Lardon (2014) proves that the convexity property holds for Bertrand oligopoly TU-games under the α and β-approaches and shows that the core is non-empty under the γ-approach.
When technologies are transferable, Zhao (1999a) provides a necessary and sufficient condition which guarantees the convexity property in Cournot oligopoly TU-games under the α and β-approaches where the inverse demand function and cost functions are linear. Although the convexity property does not hold in general, Norde et al. (2002) show that these games are totally balanced.

Until now, no work has dealt with Bertrand oligopoly TU-games with transferable technologies. In this article, as a counterpart to this lack of interest, we study such games by following the α and β-approaches. First, we show that these games are well-defined and argue that the α and β-approaches lead to the same class of games by using Zhao's result (1999b). Although the convexity property does not always hold, we prove that it is satisfied when firms' marginal costs are not too heterogeneous. Then we show that the core of any game can be partially characterized by associating a Bertrand oligopoly TU-game derived from the most efficient technology. Such a game turns to be an efficient convex cover (Rulnick and Shapley 1997) relative to the original one. This result implies that the core is non-empty and contains a subset of payoff vectors with a symmetric geometric structure, and so easy to compute.
This article is organized as follows. Section 2 gives some basic concepts on TU-games. In Section 3 we introduce the model and argue that the α and β-approaches are equivalent. In Section 4 we provide a sufficient condition under which the convexity property holds and propose a partial characterization of the core. Section 6 gives some concluding remarks.

[2] Zhao shows that the core is non-empty for general TU-games.

2 Preliminaries

We denote by $|X|$ the cardinality of a finite set X. Given $x \in \mathbb{R}$, $\lceil x \rceil$ is the smallest integer greater than or equal to x. A point $x \in \mathbb{R}^n$ is a convex combination of points $x_1, \ldots, x_k \in \mathbb{R}^n$, $k \in \mathbb{N}$, if there exist non-negative real numbers $\alpha_1, \ldots, \alpha_k$ with $\sum_{i=1}^k \alpha_i = 1$ such that $x = \sum_{i=1}^k \alpha_i x_i$. Given a finite set X, conv(X) denotes the convex hull of X, i.e. the set of points $x \in \mathbb{R}^n$ which are a convex combination of some points in X.

Let $N = \{1, \ldots, n\}$ be a fixed and finite **set of players**. We denote by 2^N the power set of N and call a subset $S \in 2^N$, $S \neq \emptyset$, a **coalition**. The **size** $s = |S|$ of coalition S is the number of players in S. A **TU-game** on N is a **set function** $v : 2^N \longrightarrow \mathbb{R}$ with the convention that $v(\emptyset) = 0$, which assigns a real number $v(S) \in \mathbb{R}$ to every coalition $S \in 2^N$. The number $v(S)$ is the worth of coalition S. We denote by G^N the **set of TU-games** on N where v is a representative element of G^N.

In a TU-game $v \in G^N$, every player $i \in N$ may receive a **payoff** $x_i \in \mathbb{R}$. A vector $x = (x_1, \ldots, x_n)$ is a **payoff vector**. A payoff vector $x \in \mathbb{R}^n$ is **acceptable** if for every coalition $S \in 2^N$, $\sum_{i \in S} x_i \geq v(S)$, i.e. the payoff vector provides a total payoff to the members of coalition S that is at least as great as its worth. A payoff vector $x \in \mathbb{R}^n$ is **efficient** if $\sum_{i \in N} x_i = v(N)$, i.e. the payoff vector provides a total payoff to all the players that is equal to the worth of the grand coalition N. The **core** $C(v) \subseteq \mathbb{R}^n$ of a TU-game $v \in G^N$ is the set of payoff vectors that are both acceptable and efficient, i.e.

$$C(v) = \Big\{ x \in \mathbb{R}^n : \forall S \in 2^N,\ \sum_{i \in S} x_i \geq v(S) \text{ and } \sum_{i \in N} x_i = v(N) \Big\}.$$

Given a payoff vector in the core, the grand coalition N forms and distributes its worth as payoffs to its members in such a way that any coalition cannot contest this sharing by breaking off from the grand coalition.

A TU-game $v \in G^N$ has a non-empty core if and only if it is balanced (Bondareva 1963 and Shapley 1967).

A **permutation** on the set of players N is a bijection $\sigma : N \longrightarrow N$ which assigns rank number $\sigma(i) \in N$ to any player $i \in N$. We denote by Π_N the set of all $n!$ permutations. For every permutation $\sigma \in \Pi_N$, we denote by $S^{\sigma,i} = \{j \in N : \sigma(j) \leq \sigma(i)\}$ the set of predecessors of i with respect to σ, including i himself. Given a TU-game $v \in G^N$ and a permutation $\sigma \in \Pi_N$, the **marginal vector** $m^\sigma(v) \in \mathbb{R}^n$ is defined as

$$\forall i \in N,\ m_i^\sigma(v) = v(S^{\sigma,i}) - v(S^{\sigma,i} \backslash \{i\})$$

and assigns to player i his marginal contribution to the worth of the coalition consisting of all his predecessors with respect to σ. A TU-game $v \in G^N$ is **convex** if

$$\forall i, j \in N,\ \forall S \in 2^{N \backslash \{i,j\}},\ v(S \cup \{i\}) - v(S) \leq v(S \cup \{i,j\}) - v(S \cup \{j\}) \quad (1)$$

A TU-game $v \in G^N$ is convex if and only if $m^\sigma(v) \in C(v)$ for all $\sigma \in \Pi_N$ (Shapley 1971 and Ichiishi 1981). The core $C(v)$ is then equal to the convex hull of the set of all the marginal vectors, i.e. $C(v) = \text{conv}(\{m^\sigma(v) : \sigma \in \Pi_N\})$.

3 Bertrand oligopoly TU-games with transferable technologies

In this section, we first define a Bertrand oligopoly situation from which we associate a normal form Bertrand oligopoly game. After formalizing the possibility for cooperating firms to transfer their technologies, we then convert the normal form Bertrand oligopoly game into a Bertrand oligopoly TU-game by using the α and β-approaches (Aumann 1959).

First, a **Bertrand oligopoly situation** is described by the tuple $(N, (D_i, C_i)_{i\in N})$ where $N = \{1, 2, \ldots, n\}$ is the finite **set of firms**, $D_i : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$, $i \in N$, is firm i's **demand function** and $C_i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $i \in N$, is firm i's **cost function**. Throughout this article, we assume that

(a) the demand system is Shubik's (1980), i.e.

$$\forall i \in N,\ D_i(p_1, \ldots, p_n) = V - p_i - r\left(p_i - \frac{1}{n}\sum_{j\in N} p_j\right),$$

where p_j is the price charged by firm j, $V \in \mathbb{R}_+$ is the intercept of demand and $r \in \mathbb{R}_{++}$ is the substitutability parameter.[3] The quantity demanded of firm i's brand depends on its price and on the difference between its price and the average price in the industry;

(b) every firm operates at a linear marginal cost $c_i \in \mathbb{R}_+$, i.e.

$$\forall i \in N,\ C_i(q_i) = c_i q_i,$$

where $q_i \in \mathbb{R}_+$ is the quantity demanded of firm i's brand.

Without loss of generality we assume that the firms are ranked according to their marginal costs, i.e. $c_1 \leq c_2 \leq \ldots \leq c_n$. Given assumptions (a) and (b), a Bertrand oligopoly situation is summarized by the 4-tuple $(N, V, r, (c_i)_{i\in N})$.

Then, corresponding to the Bertrand oligopoly situation $(N, V, r, (c_i)_{i\in N})$, the associated **normal form Bertrand oligopoly game** $(N, (X_i, \pi_i)_{i\in N})$ is defined as

1. the set of firms is $N = \{1, \ldots, n\}$;

[3] When r approaches zero, products become unrelated, and when r approaches infinity, products become perfect substitutes.

2. for every $i \in N$, the **individual strategy set** is $X_i = \mathbb{R}_+$ where $p_i \in X_i$ represents the price charged by firm i;

3. the **set of strategy profiles** is $X_N = \prod_{i\in N} X_i$ where $p = (p_i)_{i\in N}$ is a representative element of X_N; for every $i \in N$, the **individual profit function** $\pi_i : X_N \longrightarrow \mathbb{R}$ is defined as

$$\pi_i(p) = D_i(p)(p_i - c_i).$$

Since technologies are transferable, cooperating firms are allowed to produce according to the cheapest technology available in the coalition. As a consequence, for every coalition $S \in 2^N$ it holds that

(c) every cooperating firm operates at the **linear marginal cost of the most efficient firm** in S denoted by $\underline{c}_S \in \mathbb{R}_+$ where $\underline{c}_S = \min\{c_i : i \in S\}$.

We denote by $X_S = \prod_{i\in S} X_i$ the **coalition strategy set** of coalition $S \in 2^N$ and $X_{N\setminus S} = \prod_{i\in N\setminus S} X_i$ the **set of outsiders' strategy profiles** for which $p_S = (p_i)_{i\in S}$ and $p_{N\setminus S} = (p_i)_{i\in N\setminus S}$ are the representative elements respectively. For every coalition $S \in 2^N$, the **coalition profit function** $\pi_S : X_S \times X_{N\setminus S} \longrightarrow \mathbb{R}$ is defined as

$$\pi_S(p_S, p_{N\setminus S}) = \sum_{i\in S} D_i(p)(p_i - \underline{c}_S).$$

Finally, we convert the normal form Bertrand oligopoly game $(N, (X_i, \pi_i)_{i\in N})$ into a Bertrand oligopoly TU-game. As discussed in the introduction, Aumann (1959) suggested two approaches of converting a normal form game into a cooperative game called game in α and β-characteristic function form.
Given the normal form Bertrand oligopoly game $(N, (X_i, \pi_i)_{i\in N})$, the α and β-characteristic functions are defined for every coalition $S \in 2^N$ as

$$v_\alpha(S) = \max_{p_S\in X_S} \min_{p_{N\setminus S}\in X_{N\setminus S}} \pi_S(p_S, p_{N\setminus S}) \tag{2}$$

and

$$v_\beta(S) = \min_{p_{N\setminus S}\in X_{N\setminus S}} \max_{p_S\in X_S} \pi_S(p_S, p_{N\setminus S}) \tag{3}$$

respectively. We denote by $G_o^N \subseteq G^N$ the **set of Bertrand oligopoly TU-games with transferable technologies**.

The following proposition states that the β-characteristic function is well-defined in the framework of Bertrand oligopoly TU-games with transferable technologies. For notational convenience, for any $s \in \{0, \ldots, n\}$ we let $\delta_s = 1 + r(n - s)/n$.

Proposition 3.1 *Let $(N, (X_i, \pi_i)_{i\in N})$ be a normal form Bertrand oligopoly game. Then, for every coalition $S \in 2^N$ it holds that*

$$v_\beta(S) = \pi_S(\overline{p}_S, \overline{p}_{N\setminus S}),$$

where $(\overline{p}_S, \overline{p}_{N\setminus S}) \in X_S \times X_{N\setminus S}$ is given by

$$\forall i \in S,\ \overline{p}_i = \max\{\underline{c}_S, (V + \delta_s \underline{c}_S)/(2\delta_s)\} \tag{4}$$

and

$$\sum_{j\in N\setminus S} \overline{p}_j = \max\{0, (n/r)(\delta_s \underline{c}_S - V)\} \tag{5}$$

Proof: When all the firms operate at a constant and identical marginal cost $c \in \mathbb{R}_+$, Lardon (2014) solves the optimization problems defined in (3). In our model, although firms operate at possibly distinct marginal costs, we argue that for every coalition $S \in 2^N$, the worth $v_\beta(S)$ can be determined in a similar way. Given any coalition $S \in 2^N$, note that the optimization problems defined in (3) only concern the coalition profit function π_S. This implies that outsiders' marginal costs don't count in the computation of the worth $v_\beta(S)$. Moreover, since technologies are transferable cooperating firms in S operate at the same constant and identical marginal cost $\underline{c}_S \in \mathbb{R}_+$. Thus, we refer to Lardon (2014) for the technical arguments proving (4) and (5). ∎

For every coalition $S \in 2^N$, the computation of the worth $v_\beta(S)$ is consistent with the fact that the quantity demanded of cooperating firm i's brand is positive since for every firm $i \in S$, $D_i(\overline{p}) \geq 0$. Moreover, Proposition 3.1 calls for comments which will be useful for the sequel.

Remark 3.2 For every coalition $S \in 2^N$, the computation of the worth $v_\beta(S)$ results from two complementary cases:

1. if $V \leq \delta_s \underline{c}_S$, then by (4) each member $i \in S$ charges prices equal to their marginal cost, i.e. $\overline{p}_i = \underline{c}_S$, and by (5) the outsiders charge a non-negative average price, i.e. $\sum_{j\in N\setminus S} \overline{p}_j/(n-s) \geq 0$. In this case, coalition S obtains a zero profit, $v_\beta(S) = 0$.
2. if $V > \delta_s \underline{c}_S$, then by (4) each member $i \in S$ charges prices greater than their marginal cost, i.e. $\overline{p}_i > \underline{c}_S$, and by (5) the outsiders charge a zero average price, i.e. $\sum_{j\in N\setminus S} \overline{p}_j/(n-s) = 0$. In this case, coalition S obtains a positive profit $v_\beta(S) = s(V - \delta_s \underline{c}_S)^2/(4\delta_s)$.

By solving successively the two minimization and maximization problems defined in (2), we can show that the α-characteristic function is well-defined too. The proof is similar to the one in Proposition 3.1, and so it is not detailed.

For general TU-games, Zhao (1999b) shows that the α and β-characteristic functions are equal when every strategy set is compact, every utility function is continuous, and the strong separability condition is satisfied. This condition requires that the utility function of any coalition S and each of its members' utility functions have the same minimizers. We argue that Zhao's result (1999b) holds for the specific class of Bertrand oligopoly TU-games with transferable technologies. First, we compactify the strategy sets by assuming that for every firm $i \in N$, $X_i = [0, \mathbf{p}]$ where $\mathbf{p}$ is sufficiently large so that the maximization/minimization problems defined in (2) and (3) have interior solutions. Then, it is clear that every individual profit function π_i is continuous. Finally, since the demand system is symmetric and cooperating firms in any coalition S operate at the same constant and identical marginal cost $\underline{c}_S \in \mathbb{R}_+$, it follows that Bertrand oligopoly TU-games satisfy the strong separability condition.

Corollary 3.3 *Let $(N, (X_i, \pi_i)_{i \in N})$ be a normal form Bertrand oligopoly game. Then, for every coalition $S \in 2^N$ it holds that $v_\alpha(S) = v_\beta(S)$.*

This result implies that outsiders' strategy profile $\overline{p}_{N \backslash S}$ that best punishes coalition S as a first mover (α-approach) also best punishes S as a second mover (β-approach).

4 Partial characterization of the core

In this section, we first provide an example in which the convexity does not hold for a large class of Bertrand oligopoly TU-games with transferable technologies. Then, we show that the convexity property is satisfied if firms' marginal costs are not too heterogeneous. Finally, even if the core can not be always fully characterized, we identify a subset of payoff vectors with a symmetric geometric structure easy to compute.

In the general framework of TU-games, the convexity property permits to characterize the core. However, this property may fail to hold in the present model as illustrated in the following example.

Example 4.1 Consider the Bertrand oligopoly situation $(N, V, r, (c_i)_{i \in N})$ where $r = n$, $c_1 = c_2 = 0$ and for all $k \in N \backslash \{1, 2\}$, $c_k = 1$. Fix any coalition $S \in 2^{N \backslash \{1,2\}}$ such that $s = \lceil n/2 \rceil$ and assume that $V = \delta_s + 1$. We want to prove that the convexity property fails to hold for coalition S. By point 2 of Remark 3.2 the worths of relevant coalitions to test (1) are given in the following table.

T	S	$S \cup \{1\}$ and $S \cup \{2\}$	$S \cup \{1, 2\}$
$v_\beta(T)$	$\dfrac{s}{4(n-s+1)}$	$\dfrac{(s+1)(n-s+2)^2}{4(n-s)}$	$\dfrac{(s+2)(n-s+2)^2}{4(n-s-1)}$

Some elementary calculus show that $v_\beta(S \cup \{1, 2\}) - v_\beta(S \cup \{2\}) < v_\beta(S \cup \{1\}) - v_\beta(S)$ for all $n \geq 6$. □

Although the convexity property does not always hold, the following result establishes that Bertrand oligopoly TU-games with transferable technologies are convex when firms' marginal costs are not too heterogeneous.

Theorem 4.2 *Let $v_\beta \in G_o^N$ be a Bertrand oligopoly TU-game with transferable technologies corresponding to the Bertrand oligopoly situation $(N, V, r, (c_i)_{i\in N})$ where $V \geq (1+r)c_n$. Then v_β is convex if*

$$rc_1 \leq c_n - c_1 \leq \frac{V}{\sqrt{\delta_1}+\sqrt{\delta_3}}\left(\frac{1}{\sqrt{\delta_1}}+\frac{1}{\sqrt{\delta_3}}-\frac{2}{\sqrt{\delta_2}}\right) \tag{6}$$

Proof: Before dealing with the convexity of v_β, we establish three preliminary results. First, for any $s \in \{0, \ldots, n\}$ we deduce from $\delta_s \geq 1$ that $\sqrt{\delta_s} \leq \delta_s \leq \delta_0 = 1 + r$ and $1 = \delta_n \leq \sqrt{\delta_s}$. These two results with the first inequality in (6) ($\delta_0 c_1 \leq \delta_n c_n$) implies that

$$\forall s \in \{1, \ldots, n-2\},\ \sqrt{\delta_s}c_1 \leq \sqrt{\delta_{s+2}}c_n \tag{7}$$

Second, for any $s \in \{0, \ldots, n-1\}$ we have $\delta_s - \delta_{s+1} = r/n$. From the strict monotonicity and strict convexity of function $f : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ which assigns to every $x \in \mathbb{R}_{++}$ the positive real number $f(x) = 1/\sqrt{x}$, it follows that

$$\forall s \in \{1, \ldots, n-2\},\ \left(\frac{1}{\sqrt{\delta_1}}+\frac{1}{\sqrt{\delta_3}}-\frac{2}{\sqrt{\delta_2}}\right) \leq \left(\frac{1}{\sqrt{\delta_s}}+\frac{1}{\sqrt{\delta_{s+2}}}-\frac{2}{\sqrt{\delta_{s+1}}}\right) \tag{8}$$

Third, for any $s \in \{1, \ldots, n\}$ it holds that $\delta_s < 1 + r$. Hence we deduce from $V \geq (1+r)c_n$ that for every coalition $S \in 2^N$, we have $V > \delta_s \underline{c}_S$. Thus, the worth of every coalition $S \in 2^N$ is given by point 2 of Remark 3.2.

We want to prove that (1) holds in $v_\beta \in G_o^N$.
First assume that $S = \emptyset$ and consider any firms $i, j \in N$ such that $c_j \leq c_i$ without loss of generality. It follows from $\delta_2 < \delta_1$ and $c_j \leq c_i$ that $v_\beta(\{i\}) + v_\beta(\{j\}) < v_\beta(\{i, j\})$.
Then assume that $S \neq \emptyset$ and define the TU-game $\tilde{v}_\beta \in G^N$ which assigns to every coalition $S \in 2^N$, the worth $\tilde{v}_\beta(S) = (4/s)v_\beta(S) = (V - \delta_s \underline{c}_S)^2/\delta_s$ with $\tilde{v}_\beta(\emptyset) = 0$. For every coalition $S \in 2^N$, it follows from the monotonicity[4] of $\tilde{v}_\beta$ that inequality in (1) is passed from $\tilde{v}_\beta$ to v_β so that we focus on the TU-game $\tilde{v}_\beta$. Consider any firms $i, j \in N$, take any coalition $S \in 2^{N\backslash\{i,j\}}$ and without loss of generality assume that $c_j \leq c_i$. For any $\alpha_1, \alpha_2 \in \mathbb{R}_+$ and any $a, b \in \mathbb{R}$ it holds that $\alpha_1 a^2 - \alpha_2 b^2 = (\sqrt{\alpha_1}a + \sqrt{\alpha_2}b)(\sqrt{\alpha_1}a - \sqrt{\alpha_2}b)$. In the TU-game $\tilde{v}_\beta$ the marginal contribution of firm i to coalition S is then given by

[4] A TU-game $v \in G^N$ is monotonic if

$$\forall S \in 2^N,\ \forall T \in 2^N : S \subseteq T,\ v(S) \leq v(T).$$

$$
\begin{aligned}
\tilde{v}_\beta(S\cup\{i\})-\tilde{v}_\beta(S) &= \frac{(V-\delta_{s+1}\underline{c}_{S\cup\{i\}})^2}{\delta_{s+1}}-\frac{(V-\delta_s\underline{c}_S)^2}{\delta_s}\\
&= A_i(S)\times B_i(S),
\end{aligned}
$$

where

$$
A_i(S)=\Big(\frac{1}{\sqrt{\delta_{s+1}}}+\frac{1}{\sqrt{\delta_s}}\Big)V-\sqrt{\delta_{s+1}}\underline{c}_{S\cup\{i\}}-\sqrt{\delta_s}\underline{c}_S
$$

and

$$
B_i(S)=\Big(\frac{1}{\sqrt{\delta_{s+1}}}-\frac{1}{\sqrt{\delta_s}}\Big)V-\sqrt{\delta_{s+1}}\underline{c}_{S\cup\{i\}}+\sqrt{\delta_s}\underline{c}_S.
$$

We want to show that $A_i(S)\times B_i(S)\leq A_i(S\cup\{j\})\times B_i(S\cup\{j\})$.
First, we show that $A_i(S)<A_i(S\cup\{j\})$. Since $c_j\leq c_i$ we have $\underline{c}_{S\cup\{i,j\}}=\underline{c}_{S\cup\{j\}}\leq\underline{c}_{S\cup\{i\}}\leq\underline{c}_S$. Moreover, for any $s\in\{1,\dots,n-2\}$ it holds that $1\leq\delta_{s+2}<\delta_{s+1}<\delta_s$ and so $\sqrt{\delta_{s+2}}<\sqrt{\delta_{s+1}}<\sqrt{\delta_s}$. These two results permit to conclude that $A_i(S)<A_i(S\cup\{j\})$.
Then, we show that $B_i(S)\leq B_i(S\cup\{j\})$ which is equivalent with $c_j\leq c_i$ to prove the following inequality

$$
\sqrt{\delta_s}\underline{c}_S-\sqrt{\delta_{s+1}}\underline{c}_{S\cup\{i\}}-(\sqrt{\delta_{s+1}}-\sqrt{\delta_{s+2}})\underline{c}_{S\cup\{j\}}\leq V\Big(\frac{1}{\sqrt{\delta_s}}+\frac{1}{\sqrt{\delta_{s+2}}}-\frac{2}{\sqrt{\delta_{s+1}}}\Big).
$$

By (7), the second inequality of (6) and (8) it follows that

$$
\begin{aligned}
\sqrt{\delta_s}\underline{c}_S-\sqrt{\delta_{s+1}}\underline{c}_{S\cup\{i\}}-(\sqrt{\delta_{s+1}}-\sqrt{\delta_{s+2}})\underline{c}_{S\cup\{j\}} &\leq \sqrt{\delta_s}c_n-\sqrt{\delta_{s+2}}c_1\\
&\leq(\sqrt{\delta_s}+\sqrt{\delta_{s+2}})(c_n-c_1)\\
&\leq(\sqrt{\delta_1}+\sqrt{\delta_3})(c_n-c_1)\\
&\leq V\Big(\frac{1}{\sqrt{\delta_1}}+\frac{1}{\sqrt{\delta_3}}-\frac{2}{\sqrt{\delta_2}}\Big)\\
&\leq V\Big(\frac{1}{\sqrt{\delta_s}}+\frac{1}{\sqrt{\delta_{s+2}}}-\frac{2}{\sqrt{\delta_{s+1}}}\Big),
\end{aligned}
$$

which concludes the proof. ■

Note that condition (6) does not depend on all firms' marginal costs in the industry but only those of the less and most efficient firms. Moreover, for any heterogeneity level of the marginal costs c_n-c_1 satisfying $(1+r)c_1\leq c_n$, the second inequality in (6) ensures that the convexity property holds if the intercept of demand is sufficiently large.

Although the core can not be always fully characterized, a subset of payoff vectors with a symmetric geometric structure can be identified. Given a Bertrand oligopoly situation $(N, V, r, (c_i)_{i\in N})$, let $w_\beta \in G_o^N$ be the **Bertrand oligopoly TU-games with the most efficient technology** corresponding to the Bertrand oligopoly situation $(N, V, r, (\hat{c}_i)_{i\in N})$ where for all $i \in N$, $\hat{c}_i = c_1$. Since firms operate at the constant and identical marginal cost c_1, it holds that w_β is convex (Lardon 2014).

Theorem 4.3 *Let $v_\beta \in G_o^N$ be a Bertrand oligopoly TU-game with transferable technologies corresponding to the Bertrand oligopoly situation $(N, V, r, (c_i)_{i\in N})$. Then it holds that*

$$\textit{conv}(\{m^\sigma(w_\beta) : \sigma \in \Pi_N\}) \subseteq C(v_\beta),$$

where $w_\beta \in G_o^N$ is the associated Bertrand oligopoly TU-game with the most efficient technology.

Proof: We know from the convexity of w_β that $\text{conv}(\{m^\sigma(w_\beta) : \sigma \in \Pi_N\}) = C(w_\beta)$. It remains to prove that $C(w_\beta) \subseteq C(v_\beta)$. For both TU-games v_β and w_β the worths of any coalition S are given by $v_\beta(S) = \pi_S(\overline{p}_S, \overline{p}_{N\setminus S})$ and $w_\beta(S) = \pi_S(\overline{p}_{S,w_\beta}, \overline{p}_{N\setminus S,w_\beta})$. We distinguish two cases:

1. take any coalition $S \in 2^N$ such that $1 \in S$. Then, by (4) and (5) it holds that for all $i \in S$, $\overline{p}_i = \overline{p}_{i,w_\beta}$ and $\sum_{j\in N\setminus S} \overline{p}_j = \sum_{j\in N\setminus S} \overline{p}_{j,w_\beta}$ which implies that $v_\beta(S) = w_\beta(S)$. In particular, we have $v_\beta(N) = w_\beta(N)$.

2. take any coalition $S \in 2^N$ such that $1 \notin S$. We distinguish three cases:

- if $V \leq \delta_s c_1$ then by point 1 of Remark 3.2, it holds that $\overline{p}_i = \underline{c}_S$ and $\overline{p}_{i,w_\beta} = c_1$ which implies that $v_\beta(S) = w_\beta(S) = 0$.

- if $\delta_s c_1 < V \leq \delta_s \underline{c}_S$ then by points 1 and 2 of Remark 3.2, it holds that $\overline{p}_i = \underline{c}_S$ and $\overline{p}_{i,w_\beta} > c_1$ which implies that $v_\beta(S) = 0 < w_\beta(S)$.

- if $\delta_s \underline{c}_S < V$ then by point 2 of Remark 3.2, we deduce from $\delta_s c_1 \leq \delta_s \underline{c}_S$ that $v_\beta(S) \leq w_\beta(S)$.

In all cases, for every coalition $S \in 2^N \setminus \{N\}$ we have $v_\beta(S) \leq w_\beta(S)$ and $v_\beta(N) = w_\beta(N)$[5] which implies that $C(w_\beta) \subseteq C(v_\beta)$. ■

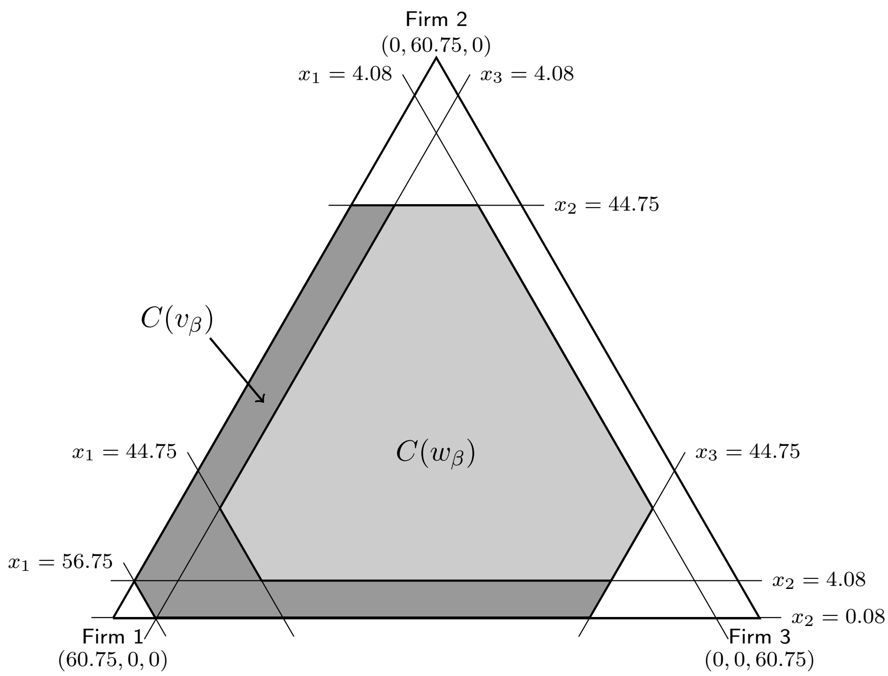
The following example shows that $C(w_\beta)$ may be a large subset of $C(v_\beta)$ and is easy to compute due to its symmetric geometric structure.

[5]Since $v_\beta(\emptyset) = w_\beta(\emptyset) = 0$, this proves that w_β is an efficient convex cover of v_β (Rulnick and Shapley 1997).

Example 4.4 Consider the Bertrand oligopoly situation $(N, V, r, (c_i)_{i\in N})$ where $N = \{1,2,3\}$, $V = 10$, $r = 3$, $c_1 = 1$, $c_2 = 3$ and $c_3 = 5$. For every coalition $S \in 2^N$, the worths $v_\beta(S)$ and $w_\beta(S)$ are given in the following table.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v_\beta(S)$	4.08	0.08	0	16	16	4	60.75
$w_\beta(S)$	4.08	4.08	4.08	16	16	16	60.75

The core $C(v_\beta)$ contains all payoff vectors $x \in \mathbb{R}^3_+$ such that $\sum_{i\in N} x_i = 60.75$, $4.08 \leq x_1 \leq 56.75$, $0.08 \leq x_2 \leq 44.75$ and $x_3 \leq 44.75$. The included core $C(w_\beta)$ contains all payoff vectors $x \in \mathbb{R}^3_+$ such that $\sum_{i\in N} x_i = 60.75$ and for all $i \in \{1,2,3\}$, $4.08 \leq x_i \leq 44.75$. The 2-simplex below represents these two core configurations.



We see that the core $C(w_\beta)$ is a large subset of the core $C(v_\beta)$. Moreover, since w_β is symmetric[6] by definition, the set of marginal vectors $\{m^\sigma(w_\beta) : \sigma \in \Pi_N\}$ is easy to compute. $\square$

5 Concluding remarks

A natural question would be whether the results in this article remain valid for convex cost functions. Insofar as the convexity property does not hold in this more general case,

[6]A TU-game $v \in G^N$ is symmetric if there exists a function $f : \mathbb{N} \longrightarrow \mathbb{R}$ such that for every coalition $S \in 2^N$, $v(S) = f(s)$.

it seems difficult to provide an appealing sufficient condition which guarantees such a property is verified.
Other approaches as the γ-approach discussed in the introduction can be considered. The computation of the worth of any coalition then requires to consider a competition setting in which cooperating firms face external ones acting individually. Unlike the α and β-approaches in which only the coalition profit function is considered, outsiders' profit functions must be taken into account. This interesting but difficult subject is left for further research.

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